Considerations on the Unruh Effect: Causality and Regularization

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Abstract

This article is motivated by the observation, that calculations of the Unruh effect based on idealized particle detectors are usually made in a way that involves integrations along the *entire* detector trajectory up to the infinitely remote *future*. We derive an expression which allows time-dependence of the detector response in the case of a non-stationary trajectory and conforms more explicitely to the principle of causality, namely that the response at a given instant of time depends only on the detectors *past* movements. On trying to reproduce the thermal Unruh spectrum we are led to an unphysical result, which we trace down to the use of the standard regularization $t \to t - i\varepsilon$ of the correlation function. By consistently employing a rigid detector of finite extension, we are led to a different regularization which works fine with our causal response function.

1 Introduction

In 1976, W. G. Unruh [1] discovered that an uniformly accelerated observer moving through *empty* Minkowski space sees a thermal radiation with a temperature directly proportional to the observers proper acceleration α:

$$T = \frac{\hbar}{2\pi ck}\,\alpha \quad .$$

This surprising and simple result, which blurs the traditional distinction between “emptiness” on the one hand and matter filled space on the other, has given rise to numerous investigations.

From the many ways of treating the Unruh effect, we concentrate on the approach originally used by Unruh and refined afterwards by DeWitt [2], which we consider the most "down to earth", namely the concept of an idealized particle detector: A pointlike quantum mechanical system with different internal energy states, which is coupled to a scalar field in its vacuum state and is moving through Minkowski space along a given trajectory $\mathsf{x}(\tau)$. A transition of the system from its ground state to an excited state will be interpreted as the detection of a particle of the corresponding energy.

The reaction of the detector to the motion $\mathsf{x}(\tau)$ is formulated via the so-called response function $F(\omega)$, which gives in essence the probability of finding the detector in an excited state of energy ω above its ground state. The usual expression for this response function can be found, for example, in [3]:

$$F(\omega) = \int\limits_{-\infty}^{\infty} \mathrm{d}\tau \int\limits_{-\infty}^{\infty} \mathrm{d}\tau' \, e^{-i\omega(\tau-\tau')} \, \langle 0|\phi(\mathsf{x}(\tau))\phi(\mathsf{x}(\tau'))|0\rangle \quad . \tag{1}$$

It consists of a twofold Fourier transform of the correlation function of the field ϕ, evaluated along the detector trajectory $\mathsf{x}(\tau)$. It may be considered a bit unfavourable, that this form of the response function does express neither time dependence nor causality, by which we mean the following: It has to be expected, that a particle detector moving through Minkowski space along a trajectory which is in general non stationary, will show different reactions at different instants of time. It is therefore desirable to have an expression for its reactions which allows an explicit time dependence, and, what is more, a time dependence of the kind, that the reaction at a given instant is influenced only by the part of the trajectory lying in the *past* of this instant. In (1) there is obviously no room for such a time-dependent and causal behaviour, because τ and τ' are integrated over and $F(\omega)$ can, by construction, never be time-dependent.

In section 2 we therefore consider the momentary transition rate $\dot{F}_\tau(\omega)$ instead of $F(\omega)$. We derive an expression for $\dot{F}_\tau(\omega)$ which depends in a manifestly causal way on the proper time τ and which can be applied to non-stationary trajectories. Unfortunately, the evaluation of this response function leads to an unphysical result already in the case of hyperbolic movement.

In section 3 we briefly review parts of an article by S. Takagi [4], where he uses a correlation function different from the standard one and tries to justify this by considering the pointlike Unruh detector as the limit of a rigid and finitely extended detector. With Takagis' correlation function, our causal response function leads to the expected result in the Unruh case.

But because we consider Takagis' argumentation not entirely correct, we propose in section 4 our own correlation function, which we consider well

founded and appropriate for the problem of the moving detector. This leads on causal calculation to the Unruh effect without any difficulties.

In section 5 we show an application to a non-stationary trajectory which smoothly starts from rest and goes over to an uniform acceleration.

In the final section 6 we briefly state our results and formulate a critical remark on an often heared argument which relates the thermality of the Unruh radiation to the existence of a so-called acceleration horizon.

We use the Minkowski metric with "spacelike" signature $(-,+,+,+)$ and denote three-vectors by boldface, four-vectors by sans-serif letters. $\hbar$ and c are set equal to 1.

2 Causal transition rate

We employ as detector an idealized "atom", i.e. a quantum mechanical system with a two-dimensional state space. The Hamiltonian H_D of the detector atom is of diagonal form with respect to the basis $|0\rangle$ (ground state) and $|1\rangle$ (excited state). The energy of the ground state is 0, the energy of the excited state is denoted by ω. The detector is moving along the trajectory $t = t(\tau),\ \mathbf{x} = \mathbf{x}(\tau)$ through a massless scalar field $\phi(t, \mathbf{x})$ to which it is coupled by the interaction $\mu(\tau)\phi(\tau)$. $\mu(\tau)$ is the detectors monopole moment, $\phi(\tau) := \phi(t(\tau), \mathbf{x}(\tau))$ the scalar field at the detectors position at time τ. The complete Hamiltonian is $H = H_D + H_F + \mu\phi$, where the explicit forms of H_D, H_F and μ can be left unspecified.

We now ask the following question: Suppose an energy measurement at time τ_0 has given the value 0 for the detector atom and the vacuum state for the scalar field. Then, what is the probability of finding the excited state of the detector, i.e. the energy value ω, in a later measurement at time $\tau > \tau_0$ if the detector has in the meantime been moving along a given trajectory? We do the calculation in the interaction picture, in which both, states and operators, are time-dependent. The field operator has the form

$$\phi(t, \mathbf{x}) = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2\omega(\mathbf{k})} \left(a(\mathbf{k}) e^{-i(\omega(\mathbf{k})t - \mathbf{k}\mathbf{x})} + a^\dagger(\mathbf{k}) e^{i(\omega(\mathbf{k})t - \mathbf{k}\mathbf{x})} \right) \tag{2}$$

where the creation- and annihilation-operators obey the covariant commutation relations

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}') \quad ,$$

and $\omega(\mathbf{k}) = |\mathbf{k}|$. The time evolution of the states is governed by

$$i \frac{\mathrm{d}}{\mathrm{d}\tau} |\psi(\tau)\rangle = \mu(\tau)\phi(\tau)|\psi(\tau)\rangle$$

where

$$\mu(\tau) = e^{iH_D\tau}\mu(0)e^{-iH_D\tau} \quad .$$

At τ_0 the system is in the product state $|\psi(\tau_0)\rangle = |0\rangle|0\rangle$, i.e. both the detector and the field are in their ground state. Then to the first order of perturbation theory

$$|\langle 1\varphi|\psi(\tau)\rangle|^2 = |\langle 1|\mu(0)|0\rangle|^2 \int\limits_{\tau_0}^{\tau} \mathrm{d}\tau' \int\limits_{\tau_0}^{\tau} \mathrm{d}\tau''\, e^{-i\omega(\tau'-\tau'')}\langle 0|\phi(\tau')|\varphi\rangle\langle\varphi|\phi(\tau'')|0\rangle$$

is the probability to find the detector in the state $|1\rangle$ and the field in the state $|\varphi\rangle$ at time τ. Summing over the unobserved field states and using their completeness, we arrive at

$$\sum_{|\varphi\rangle} |\langle 1\varphi|\psi(\tau)\rangle|^2 = |\langle 1|\mu(0)|0\rangle|^2 \int\limits_{\tau_0}^{\tau} \mathrm{d}\tau' \int\limits_{\tau_0}^{\tau} \mathrm{d}\tau''\, e^{-i\omega(\tau'-\tau'')}\langle 0|\phi(\tau')\phi(\tau'')|0\rangle \quad ,$$

which is the desired probability of finding the detector in its excited state at time τ. It consists of a constant, detector-specific coefficient and an integral which depends only on the field ϕ. We are interested only in the integral which we denote by $F_\tau(\omega)$, and we introduce the abbreviation $W(\tau,\tau')$ ("Wightman-function") for the correlation function $\langle 0|\phi(\tau)\phi(\tau')|0\rangle$:

$$F_\tau(\omega) = \int\limits_{\tau_0}^{\tau} \mathrm{d}\tau' \int\limits_{\tau_0}^{\tau} \mathrm{d}\tau''\, e^{-i\omega(\tau'-\tau'')} W(\tau,\tau') \quad .$$

The integration extends over a square in the τ', τ''-plane. We can write this in a more convenient form by introducing new coordinates $u = \tau'$, $s = \tau' - \tau''$ in the lower triangle $\tau'' < \tau$, and $u = \tau''$, $s = \tau'' - \tau'$ in the upper triangle $\tau'' > \tau'$. This leads to

$$F_\tau(\omega) = \int\limits_{\tau_0}^{\tau} \mathrm{d}u \int\limits_{0}^{u-\tau_0} \mathrm{d}s\ \left(e^{-i\omega s}\, W(u, u-s) \;+\; e^{+i\omega s}\, W(u-s, u)\right) \quad ,$$

which is the same as

$$F_\tau(\omega) = 2\int\limits_{\tau_0}^{\tau} \mathrm{d}u \int\limits_{0}^{u-\tau_0} \mathrm{d}s\, \mathsf{Re}\left(e^{-i\omega s}\, W(u, u-s)\right)$$

because of $W(\tau',\tau) = W^*(\tau,\tau')$. We can read off the following structure: The probability of finding the detector at time τ in a state, whose energy is different by an amount ω from the energy of the ground state, is given as a

sum of contributions from every time u between τ_0 and τ. The contribution of a certain time u is in turn given by the correlations of $\phi(u)$ with ϕ at every past time since τ_0. Therefore, the time derivative of $F_\tau(\omega)$, i.e. the transition rate $\dot{F}_\tau(\omega)$, is not determined only by the state of motion at time τ but by the whole course of the trajectory between τ_0 and τ:

$$\dot{F}_\tau(\omega) = 2 \int_0^{\tau-\tau_0} \mathrm{d}s \; \mathsf{Re}\left(e^{-i\omega s}\; W(\tau, \tau - s)\right) \quad .$$

From now on we will consider only the transition rate $\dot{F}_\tau(\omega)$ which is proportional to the number of "clicks" per second in a detector consisting of a swarm of identical detector atoms, and which is physically more accessible than the probability $F_\tau(\omega)$. (In the following we will call $\dot{F}_\tau(\omega)$ "response-function", which normally rather is the name for the probability $F_\tau(\omega)$.) Furthermore we get rid of the arbitrariliy chosen τ_0 by doing the limit $\tau_0 \to -\infty$:

$$\dot{F}_\tau(\omega) = 2 \int_0^{\infty} \mathrm{d}s \; \mathsf{Re}\left(e^{-i\omega s}\; W(\tau, \tau - s)\right) \quad . \tag{3}$$

If W is invariant under time translation, i.e. $W(\tau_1+\Delta\tau, \tau_2+\Delta\tau) = W(\tau_1, \tau_2)$ then it is an easy matter to show that

$$2 \int_0^{\infty} \mathrm{d}s \; \mathsf{Re}\left(e^{-i\omega s}\; W(\tau, \tau - s)\right) = \int_{-\infty}^{\infty} \mathrm{d}s \; e^{-i\omega s}\; W(s, 0) \quad . \tag{4}$$

That means, if W is invariant under time translation, $\dot{F}_\tau(\omega)$ is independent of τ and can formally be written as an integral over the *whole* (i.e. past and *future*) trajectory. This does not lead to any problem in the case of a stationary trajectory, because this is a "global" object anyway, for an arbitrarily small piece of such a trajectory contains all the information about its whole, in particular its future, course. But in the general case of a non-stationary trajectory, one must use (3) which is therefore the starting point of our further considerations.

Up to now, our reasoning was completely general and we did not make any use of the fact that ϕ is a scalar field of the form (2). In the infinite-dimensional case of the scalar field, it has to be taken into account that W is not a function but a distribution. This is clear from

$$\begin{aligned} \langle 0|\phi(\mathsf{x})\phi(\mathsf{x}')|0\rangle &= \frac{1}{(2\pi)^6} \langle 0| \int \frac{\mathrm{d}^3k\, \mathrm{d}^3k'}{4\omega(\mathbf{k})\omega(\mathbf{k}')}\, a(\mathbf{k})a^\dagger(\mathbf{k}')\, e^{i(\mathsf{kx}-\mathsf{k'x'})}\, |0\rangle \\ &= \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3k}{2\omega(\mathbf{k})}\, e^{i\mathsf{k}(\mathsf{x}-\mathsf{x}')} \quad , \end{aligned}$$

because the integrand is oscillatory and the integral does not converge in the classical sense. The standard way to handle this is to make the replacement $t \to t - i\varepsilon$, or, equivalent, to insert an exponential factor $e^{-|\mathbf{k}|\varepsilon}$ into the Fourier representation, which regularizes the high-frequency behavior and take the limit $\varepsilon \to 0$ at the end of the calculation, i.e.

$$\langle 0|\phi(\mathsf{x})\phi(\mathsf{x}')|0\rangle = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3 k}{2\omega(\mathbf{k})}\, e^{i\mathsf{k}(\mathsf{x}-\mathsf{x}')-|\mathbf{k}|\varepsilon} \tag{5}$$

and

$$\dot{F}_\tau(\omega) = 2\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{d}s\ \mathsf{Re}\left(e^{-i\omega s}\ W_\varepsilon(\tau,\tau-s)\right) \quad . \tag{6}$$

We will investigate later how this regularization can be done in a more physically motivated way. Accepting the $e^{-|\mathbf{k}|\varepsilon}$-factor for the moment, one arrives after a short calculation at

$$\langle 0|\phi(\mathsf{x})\phi(\mathsf{x}')|0\rangle = \frac{-1/4\pi^2}{(t-t'-i\varepsilon)^2-(\mathbf{x}-\mathbf{x}')^2} \quad . \tag{7}$$

This is the form of the correlation function which is used in the standard literature, see e.g. [3], (3.59). If the trajectory of a detector at rest, $t(\tau) = \tau,\ \mathbf{x}(\tau) = 0$ is inserted, one gets

$$\dot{F}_\tau(\omega) = -\frac{1}{2\pi^2}\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{d}s\ \mathsf{Re}\ \frac{e^{-i\omega s}}{(s-i\varepsilon)^2} = -\frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\int_{-\infty}^\infty \mathrm{d}s\ \frac{e^{-i\omega s}}{(s-i\varepsilon)^2} \quad .$$

One can evaluate this integral by contour-integration along the real axis closed by an infinite semi-circle in the upper or lower half-plane, depending on the sign of ω. The resulting value is $2\pi\omega e^{\varepsilon\omega}\Theta(-\omega)$ which leads to

$$\dot{F}_\tau(\omega) = -\frac{1}{2\pi}\,\omega\Theta(-\omega) \quad .$$

This result - independent of τ as expected - describes the fact that a detector atom at rest does not go spontaneously into its excited state ($\dot{F}_\tau(\omega) = 0\ \forall\omega > 0$), but of course it decays spontaneously into its ground state due to the coupling to the scalar field.

Now the interesting thing is to see what happens if the detector is uniformly accelerated, i.e.[1]

$$t(\tau) = \sinh(\tau), \quad x(\tau) = \cosh(\tau), \quad y(\tau) = z(\tau) = 0 \quad .$$

[1]We set the proper acceleration equal to 1 because the transition-rate for arbitrary acceleration $\alpha > 0$ can be obtained by a simple scale-transformation $\dot{F}_\tau(\omega) = \alpha\dot{F}^{(\alpha=1)}_{\alpha\tau}\left(\frac{\omega}{\alpha}\right)$.

Because this is an orbit of the Killing vector $x\partial_t + t\partial_x$ it is stationary and one expects a time-independent transition rate. Therefore it is a bit surprising to see that insertion of this trajectory into (7) leads to a $W_\varepsilon(\tau, \tau - s)$ which is *not* time-independent, i.e. not independent of τ:

$$W_\varepsilon(\tau, \tau - s) = \frac{-1/4\pi^2}{4\sinh^2\left(\frac{s}{2}\right) - \varepsilon^2 - 2i\varepsilon\left(\sinh(\tau) - \sinh(\tau - s)\right)} \quad .$$

But this in itself is not to alarming, because time independence is demanded only for the transition rate, which contains a limit $\varepsilon \to 0$, and $W_0(\tau, \tau - s)$ is clearly independent of τ. To settle the case, let us do a numerical calculation of $\dot{F}_\tau(\omega)$: The response function (6) is of the form

$$\dot{F}_\tau(\omega) \quad = \quad 2\lim_{\varepsilon \to 0} \int\limits_0^\infty \mathrm{d}s \;\left(a_\varepsilon(\tau, \tau - s)\cos\omega s + b_\varepsilon(\tau, \tau - s)\sin\omega s\right) \quad ,$$

where $a_\varepsilon := \mathsf{Re}W_\varepsilon$ and $b_\varepsilon := \mathsf{Im}W_\varepsilon$, i.e.

$$\begin{aligned} a_\varepsilon(\tau, \tau - s) \quad &= \quad -\frac{1}{4\pi^2} \frac{4\sinh^2(\frac{s}{2}) - \varepsilon^2}{\left(4\sinh^2(\frac{s}{2}) - \varepsilon^2\right)^2 + 4\varepsilon^2(\sinh\tau - \sinh(\tau - s))^2} \\ b_\varepsilon(\tau, \tau - s) \quad &= \quad -\frac{1}{4\pi^2} \frac{2\varepsilon(\sinh\tau - \sinh(\tau - s))}{\left(4\sinh^2(\frac{s}{2}) - \varepsilon^2\right)^2 + 4\varepsilon^2(\sinh\tau - \sinh(\tau - s))^2} \quad . \end{aligned}$$

In order to calculate the transition rate for a given time τ, we thus have to evaluate the cosine- or sine-transform of $a_\varepsilon(\tau, \tau - s)$ and $b_\varepsilon(\tau, \tau - s)$ with respect to s at fixed τ. The numerical results for seven different times $\tau = 0, \pm 1, \pm 2, \pm 4$ are shown in figure 1. The lower curve corresponds to $\tau = -4$, higher curves to increasingly later times. Thus our suspicion is confirmed that the τ-dependence of the correlation function leads to a τ-dependent transition rate, which is unacceptable in the case at hand. Furthermore, the results are problematic for $\tau < 0$ because $\dot{F}_\tau(\omega)$ assumes negative values in this cases.

At this point, the first question which comes to mind is to ask the reliability of the numerical scheme. Fortunately, an exact calculation is possible at least in the two special cases of $\tau = 0$ or $\omega = 0$. In the case $\tau = 0$ the transition rate can be evaluated by the method of contour-integration. This results in

$$\dot{F}_0(\omega) = \frac{1}{2\pi} \frac{\omega}{e^{2\pi\omega} - 1} \quad ,$$

which is indeed the usual Unruh spectrum. This is exactly in agreement with the numerical result for $\tau = 0$. In the other case, $\omega = 0$, the integral can be evaluated exactly with the help of Maple, which leads to

$$\dot{F}_\tau(0) = \frac{1 + e^{2\tau}(2\tau - 1)}{2\pi^2(e^{2\tau} - 1)^2} \quad .$$

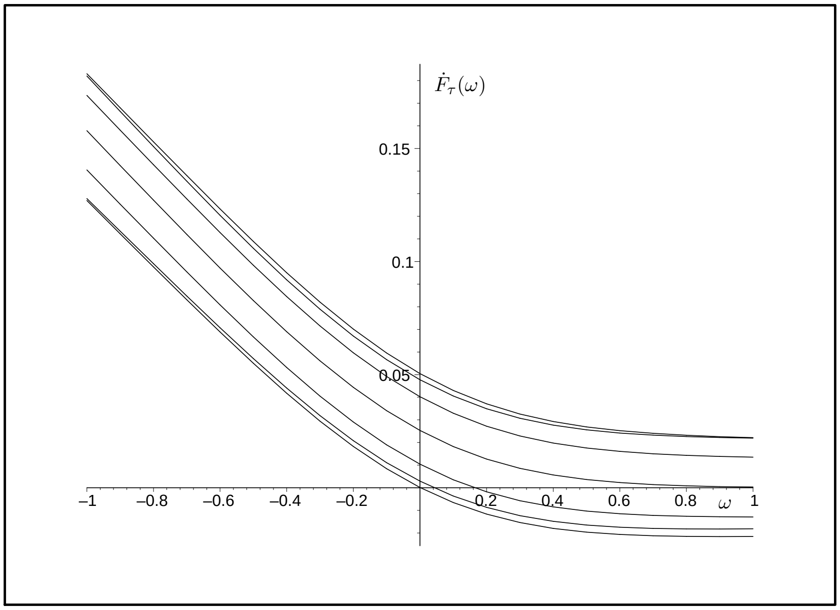

Figure 1: Numerical results for $\dot{F}_\tau(\omega)$

In this way we get on the one hand a rigorous proof that $\dot{F}_\tau(\omega)$ is indeed τ-dependent if the calculation is based on (7). On the other hand this exact result permits a further test of the numerical calculation. For example, the numerics tells us that $\dot{F}_1(0) = 0.0104...$, and indeed

$$\dot{F}_1(0) = \frac{1+e^2}{2\pi^2(e^2-1)^2} = 0.0104... \quad .$$

For any other checked $\dot{F}_\tau(0)$ there is an equally good agreement between the numerical prediction and the analytic result. This gives us confidence in the correctness of the numerical calculation in the general case.

We are thus confronted with the following situation: The application of our causal formulation of the detector response to the case of uniformly accelerated motion lead to a unphysical, time-dependent result. But we see no reason to believe that it is impossible to derive the Unruh effect consistently in an explicitly causal way, or, more generally, to formulate a causal response-function which can be applied to non-stationary trajectories. Therefore, we will seek the error not in the use of our causal response function (6), but rather suspect that the correlation function (7) and the $e^{-|\mathbf{k}|\varepsilon}$-regularization of the integral representation (5) is not suitable for the problem of the moving detector. In the following sections we will consider a finite, rigid detector and show that one is led in a natural way to a correlation function which is

different from (7)[2]. We use the article [4] of S. Takagi as a model, even if we come to a different result. The new correlation function leads upon causal calculation to the desired Unruh result without any difficulty, and it seems to be superior to (7) in other ways.

3 Correlation function a la Takagi

In [4] Takagi criticizes the ad-hoc regularization of the correlation function (5) by the introduction of the factor $e^{-|\mathbf{k}|\varepsilon}$ into its Fourier representation or by the equivalent prescription $t \to t - i\varepsilon$. Rather he would like to understand the origin of the regulator as a consequence of the finite extension of the detector. Therefore he uses instead of the ultralocal $\phi(\tau) = \phi(\mathsf{x}(\tau))$ the "smeared" version

$$\phi(\tau) = \int \mathrm{d}^3\xi \; f(\boldsymbol{\xi}) \; \phi(\mathsf{x}(\tau, \boldsymbol{\xi})) \quad ,$$

where $f(\boldsymbol{\xi})$ is a weight function which is normalized, symmetric under rotations and concentrated around the origin. By choosing $f(\boldsymbol{\xi}) = \delta(\boldsymbol{\xi})$ one recovers the pointlike detector; we will consider a family of finite detectors $f_\varepsilon(\boldsymbol{\xi})$ instead with $\lim_{\varepsilon\to 0} f_\varepsilon(\boldsymbol{\xi}) = \delta(\boldsymbol{\xi})$. By taking the limit only at the end of the calculation, we thus model an "infinitesimal" instead of a pointlike detector.

Of special importance is the function $\mathsf{x}(\tau, \boldsymbol{\xi})$, so we have to say a little bit about it. It describes the transformation from the usual Minkowski coordinates to the Fermi-Walker coordinates τ and $\boldsymbol{\xi} := (\xi, \eta, \zeta)$ associated with the trajectory[3] $\mathsf{x} = \mathsf{x}(\tau)$:

$$\begin{aligned} t(\tau, \boldsymbol{\xi}) &= t(\tau) + \xi \dot{x}(\tau) \\ \mathbf{x}(\tau, \boldsymbol{\xi}) &= \begin{pmatrix} x(\tau) + \xi \dot{t}(\tau) \\ \eta \\ \zeta \end{pmatrix} \quad . \end{aligned} \tag{8}$$

The geometrical meaning of these coordinates is the following: To every point $\mathsf{x}(\tau) = (t(\tau), x(\tau), 0, 0)$ on the trajectory belongs its simultaneity, i.e. the hyperplane orthogonal to the four-velocity $\mathsf{u}(\tau) = (\dot{t}(\tau), \dot{x}(\tau), 0, 0)$ in this point. This three-dimensional space consists of all events which are simultaneous to $\mathsf{x}(\tau)$, where simultanity is judged from the co-moving inertial frame. If we attach orthonormal basis vectors $\mathsf{e}_\xi(\tau) = (\dot{x}(\tau), \dot{t}(\tau), 0, 0)$, $\mathsf{e}_\eta(\tau) = (0, 0, 1, 0)$, $\mathsf{e}_\zeta(\tau) = (0, 0, 0, 1)$ to every such hyperplane, we can characterize every event x (in a neighborhood of the trajectory) by (τ, ξ, η, ζ): First, we

[2] In [3] the Unruh spectrum is derived from (7) only by the use of a not entirely convincing calculation in which "a positive function of τ, τ' is absorbed into ε".

[3] For the moment, we restrict ourselves to trajectories which run along the x-axis, i.e. $y = z = 0$

fix τ by the condition that the given spacetime point lies in the simultanity of the trajectory point $\mathsf{x}(\tau)$. Then $\mathsf{x} - \mathsf{x}(\tau)$ is a linear combination of $\mathsf{e}_\xi(\tau), \mathsf{e}_\eta(\tau), \mathsf{e}_\zeta(\tau)$, and we can read off (ξ, η, ζ) as the corresponding coefficients. That is[4]

$$\mathsf{x}(\tau) + \xi \mathsf{e}_\xi(\tau) + \eta \mathsf{e}_\eta(\tau) + \zeta \mathsf{e}_\zeta(\tau) = t\mathsf{e}_t + x\mathsf{e}_x + y\mathsf{e}_y + z\mathsf{e}_z \quad ,$$

which is nothing else but (8). Expressed by the coordinates $(\tau, \boldsymbol{\xi})$, the Minkowski metric assumes the form

$$\mathrm{d}s^2 = -\left(1 \,+\, 2(\dot{t}\ddot{x} - \dot{x}\ddot{t})\xi \,+\, (\ddot{x}^2 - \ddot{t}^2)\xi^2\right) \mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{\xi}^2 \quad .$$

Obviously, every simultanity $\tau = \text{const.}$ carries the same cartesian three-metric. Because $\phi(\tau)$ originates from an integration over the τ-simultanity, the τ-independent shape function $f(\boldsymbol{\xi})$ ensures that this construction corresponds indeed to a *rigid* detector - where rigidity means that its three-geometry as seen from its own momentary rest system is unchanged in the course of proper time. In the case of hyperbolic motion we have for example

$$\begin{aligned} t(\tau, \boldsymbol{\xi}) &= (1+\xi)\sinh\tau \\ \mathbf{x}(\tau, \boldsymbol{\xi}) &= \begin{pmatrix} (1+\xi)\cosh\tau \\ \eta \\ \zeta \end{pmatrix} \end{aligned} \quad ,$$

and the metric

$$\mathrm{d}s^2 = -(1+\xi)^2 \mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{\xi}^2 \quad ,$$

i.e. the well known transformation to Rindler coordinates, cf. [5].

Now by linearizing $\mathsf{x}(\tau, \boldsymbol{\xi})$ around $\boldsymbol{\xi} = 0$

$$\mathsf{x}(\tau, \boldsymbol{\xi}) \approx \mathsf{x}(\tau) \,+\, \boldsymbol{\xi}\, \frac{\partial \mathsf{x}(\tau, \boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\Big|_{\boldsymbol{\xi}=0} \;=:\; \mathsf{x}(\tau) \,+\, \boldsymbol{\xi} \mathsf{X}_{\boldsymbol{\xi}}(\tau)$$

Takagi finds

$$\phi(\tau) = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3 k}{2|\mathbf{k}|} \left(a(\mathbf{k}) e^{-i\mathsf{k}\mathsf{x}(\tau)} \tilde{f}(\mathbf{k}; \tau) \,+\, a^\dagger(\mathbf{k}) e^{i\mathsf{k}\mathsf{x}(\tau)} \tilde{f}^*(\mathbf{k}; \tau) \right) \quad ,$$

where

$$\tilde{f}(\mathbf{k}; \tau) := \int \mathrm{d}^3\xi \; f(\boldsymbol{\xi}) \; e^{-i\mathsf{k}\,\boldsymbol{\xi}\mathsf{X}_{\boldsymbol{\xi}}(\tau)}$$

converges to 0 for $|\mathbf{k}| \to \infty$, which makes $\phi(\tau)$ a well-defined operator. He now chooses for $f(\boldsymbol{\xi})$ the family of functions

$$f_\varepsilon(\boldsymbol{\xi}) = \frac{1}{\pi^3} \frac{\varepsilon^3}{(\xi^2 + \varepsilon^2)(\eta^2 + \varepsilon^2)(\zeta^2 + \varepsilon^2)} \quad ,$$

[4] τ is used with two different meanings: As proper time of the trajectory, and as spacetime coordinate. This should not lead to any confusion.

which converges to the three-dimensional δ-function in the limit $\varepsilon \to 0$. Then $\tilde{f}(\mathbf{k};\tau)$ can be calculated explicitly:

$$\tilde{f}_\varepsilon(\mathbf{k};\tau) = e^{-\varepsilon|\mathsf{k}\mathsf{X}_\xi(\tau)|-\varepsilon|\mathsf{k}\mathsf{X}_\eta(\tau)|-\varepsilon|\mathsf{k}\mathsf{X}_\zeta(\tau)|} \quad . \tag{9}$$

Takagi now claims that this complicated expression can be effectively replaced by the simpler

$$\tilde{f}_\varepsilon(\mathbf{k};\tau) = e^{-|\mathbf{k}|\varepsilon/2} \quad , \tag{10}$$

which does not depend on τ any more. Accepting this for the moment, one arrives after a few lines at

$$\langle 0|\phi(\tau)\phi(\tau')|0\rangle = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2|\mathbf{k}|} e^{i\mathsf{k}(\mathsf{x}-\mathsf{x}')-|\mathbf{k}|\varepsilon} \quad . \tag{11}$$

Obviously, the finite extension $\varepsilon > 0$ of the detector manifests itself in a regularization of the integral, as desired. Takagi now writes this in four-dimensional form

$$\langle 0|\phi(\tau)\phi(\tau')|0\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^4k\ \Theta(k^0)\delta(\mathsf{k}^2)e^{i\mathsf{k}(\mathsf{x}-\mathsf{x}')-k^0\varepsilon} \quad , \tag{12}$$

and claims this form to be manifestly Lorentz invariant. He now uses this Lorentz invariance to derive

$$\langle 0|\phi(\tau)\phi(\tau')|0\rangle = \frac{-1/4\pi^2}{\left(\mathrm{sgn}(t-t')\sqrt{-(\mathsf{x}-\mathsf{x}')^2} - i\varepsilon\right)^2} \quad . \tag{13}$$

By inserting the hyperbolic trajectory and using contour-integration, it follows indeed that

$$\dot{F}_\tau(\omega) = \frac{1}{2\pi}\frac{\omega}{e^{2\pi\omega}-1} \quad .$$

Even if this is the desired result, we have some reservations concerning the derivation. First, the claim that the integral (12) is Lorentz invariant holds only if $\varepsilon = 0$. But in our case $\varepsilon > 0$ the factor $e^{-k^0\varepsilon}$ destroys this invariance, which makes the evaluation of the integral obsolete. Indeed, straightforward evaluation of (11) leads exactly to the standard form (7) of the correlation function and not to (13). Therefore, if Takagi's derivation were correct up to (11) this would show that taking into account the finite extension of the detector does not solve the problem which we encountered in the last section. We now suppose, that, contrary to Takagi's assertion, it is *not* allowed to replace (9) by (10). Indeed this replacement eliminates the dependence on τ and on the trajectory, and it is not clear that this leaves the final result unchanged. Therefore we will try to work out Takagi's idea more carefully in the next section.

4 Correlation function of an infinitesimal rigid detector

We now use the monopole interaction with a "smeared" field-operator

$$\phi(\tau) = \int \mathrm{d}^3\xi \; f(\boldsymbol{\xi}) \; \phi(\mathsf{x}(\tau, \boldsymbol{\xi})) \tag{14}$$

instead of $\phi(\tau) = \phi(\mathsf{x}(\tau))$. The correlation function thus assumes the form:

$$\begin{aligned}\langle 0|\phi(\tau)\phi(\tau')|0\rangle &= \frac{1}{(2\pi)^6}\,\langle 0| \int \mathrm{d}^3\xi \; f(\boldsymbol{\xi}) \int \mathrm{d}^3\xi' \; f(\boldsymbol{\xi}') \\ &\int \frac{\mathrm{d}^3k \; \mathrm{d}^3k'}{4\omega(\mathbf{k})\omega(\mathbf{k}')} a(\mathbf{k})a^\dagger(\mathbf{k}') \; e^{i\mathsf{k}\mathsf{x}(\tau,\boldsymbol{\xi})} e^{-i\mathsf{k}'\mathsf{x}(\tau',\boldsymbol{\xi}')} \; |0\rangle \quad .\end{aligned}$$

Using the commutation relations of the a and $a^\dagger$, $a|0\rangle = 0$ and $\langle 0|0\rangle = 1$ we can do the $\mathbf{k}'$-integration and find

$$\langle 0|\phi(\tau)\phi(\tau')|0\rangle = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3k}{2\omega(\mathbf{k})} \; g(\mathbf{k};\tau) g^*(\mathbf{k};\tau')$$

where we have introduced $g(\mathbf{k};\tau) := \int \mathrm{d}^3\xi \; f(\boldsymbol{\xi}) e^{i\mathsf{k}\mathsf{x}(\tau,\boldsymbol{\xi})}$. In order to find the correlation function of a detector with shape $f(\boldsymbol{\xi})$ and trajectory $\mathsf{x}(\tau)$ we thus have to determine the transformation to Fermi coordinates $\mathsf{x}(\tau, \boldsymbol{\xi})$, from that the function $g(\mathbf{k};\tau)$, and finally $\langle 0|\phi(\tau)\phi(\tau')|0\rangle$. This amounts to six integrations, so one would expect difficult computations for even the simplest trajectories. In fact we will see, that by choosing a suitable shape $f(\boldsymbol{\xi})$ of the detector, one arrives at an explicit form of the correlation-function for any arbitrary trajectory.

Let us first consider the simple example of a detector which is not moving at all, and whose shape is given by a function from the family

$$f_\varepsilon(\boldsymbol{\xi}) = \frac{1}{\pi^2} \frac{\varepsilon}{(\boldsymbol{\xi}^2 + \varepsilon^2)^2} \quad , \quad \varepsilon > 0 \quad . \tag{15}$$

The $f_\varepsilon(\boldsymbol{\xi})$ are invariant under rotations, satisfy $\int \mathrm{d}^3\xi \; f_\varepsilon(\boldsymbol{\xi}) = 1 \quad \forall \varepsilon > 0$ and have the scaling property $f_\varepsilon(\boldsymbol{\xi}) = \varepsilon^{-3} f_1(\boldsymbol{\xi}/\varepsilon)$. This means, they approximate the three-dimensional δ-function. The parameter ε has the physical dimension of a length and indicates the extension of the detector: $f_\varepsilon(\boldsymbol{\xi})$ is negligible outside $\boldsymbol{\xi}^2 < \varepsilon^2$. By taking the limit $\varepsilon \to 0$ at the end of the computation one can model an infinitesimal detector. The transformation to Fermi coordinates is trivial in the case of a detector at rest: $t(\tau, \boldsymbol{\xi}) = \tau, \; \mathbf{x}(\tau, \boldsymbol{\xi}) = \boldsymbol{\xi}$. This leads to

$$g_\varepsilon(\mathbf{k};\tau) = \int \mathrm{d}^3\xi \; f_\varepsilon(\boldsymbol{\xi}) e^{i\mathsf{k}\mathsf{x}(\tau,\boldsymbol{\xi})} = e^{-i|\mathbf{k}|\tau} e^{-\varepsilon|\mathbf{k}|} \quad .$$

From that:

$$\begin{aligned}\langle 0|\phi(\tau)\phi(\tau')|0\rangle &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3k}{2|\mathbf{k}|}\, g_\varepsilon(\mathbf{k};\tau)g_\varepsilon^*(\mathbf{k};\tau') \\ &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, \frac{e^{-i|\mathbf{k}|(\tau-\tau'-i\varepsilon)}}{2|\mathbf{k}|} \\ &= \frac{-1/4\pi^2}{(\tau-\tau'-i\varepsilon)^2} \quad . \qquad (16)\end{aligned}$$

(We have replaced 2ε by ε.) Using the same contour-integration as before in section 2 it follows

$$\dot{F}_\tau(\omega) = -\frac{1}{2\pi}\omega\Theta(-\omega) \quad .$$

So in this simple case we get the same result as before in section 2 where we inserted the trajectory into the Wightman function (7). But we now see that the high-frequency regulator $e^{-\varepsilon|\mathbf{k}|}$, which was put in by hand in section 2, results from the finite extension of the rigid detector and thus acquires a physically meaningful interpretation.

Let us now consider an arbitrary trajectory $\mathsf{x} = \mathsf{x}(\tau)$. The transformation to Fermi coordinates is

$$\mathsf{x}(\tau,\boldsymbol{\xi}) = \mathsf{x}(\tau) + \xi\mathsf{e}_\xi(\tau) + \eta\mathsf{e}_\eta(\tau) + \zeta\mathsf{e}_\zeta(\tau) \quad ,$$

where $\mathsf{e}_\xi(\tau), \mathsf{e}_\eta(\tau), \mathsf{e}_\zeta(\tau)$ together with $\mathsf{u}(\tau) = \dot{\mathsf{x}}(\tau)$ form an orthonormal basis which is Fermi-Walker transported along $\mathsf{x}(\tau)$. We will use the same detector-shape (15) as in the case of the detector at rest. Let us start with the calculation of $g_\varepsilon(\mathbf{k};\tau)$:

$$g_\varepsilon(\mathbf{k};\tau) = \int \mathrm{d}^3\xi\; f_\varepsilon(\boldsymbol{\xi}) e^{i\mathsf{k}\mathsf{x}(\tau,\boldsymbol{\xi})} = e^{i\mathsf{k}\mathsf{x}(\tau)} \int \mathrm{d}^3\xi\; f_\varepsilon(\boldsymbol{\xi}) e^{i(\xi\mathsf{k}\mathsf{e}_\xi(\tau)+\eta\mathsf{k}\mathsf{e}_\eta(\tau)+\zeta\mathsf{k}\mathsf{e}_\zeta(\tau))} \quad .$$

We now introduce the abbreviation $\bar{\mathbf{k}} := (\mathsf{k}\mathsf{e}_\xi(\tau), \mathsf{k}\mathsf{e}_\eta(\tau), \mathsf{k}\mathsf{e}_\zeta(\tau))$. It is easy to do the integration by using spherical coordinates in $\boldsymbol{\xi}$-space:

$$\int \mathrm{d}^3\xi\; f_\varepsilon(\boldsymbol{\xi})\; e^{i\bar{\mathbf{k}}\boldsymbol{\xi}} = e^{-\varepsilon|\bar{\mathbf{k}}|} \quad .$$

That means

$$g_\varepsilon(\mathbf{k};\tau) = e^{i\mathsf{k}\mathsf{x}(\tau)} e^{-\varepsilon\sqrt{(\mathsf{k}\mathsf{e}_\xi(\tau))^2+(\mathsf{k}\mathsf{e}_\eta(\tau))^2+(\mathsf{k}\mathsf{e}_\zeta(\tau))^2}} \quad .$$

Because k is a lightlike four-vector, the sum of squares under the square root is a complete square: By multiplying the identity

$$\mathsf{k} = -(\mathsf{k}\mathsf{u})\mathsf{u} + (\mathsf{k}\mathsf{e}_\xi)\mathsf{e}_\xi + (\mathsf{k}\mathsf{e}_\eta)\mathsf{e}_\eta + (\mathsf{k}\mathsf{e}_\zeta)\mathsf{e}_\zeta$$

with k we get

$$(\mathsf{k}\mathsf{e}_\xi(\tau))^2 + (\mathsf{k}\mathsf{e}_\eta(\tau))^2 + (\mathsf{k}\mathsf{e}_\zeta(\tau))^2 = (\mathsf{k}\mathsf{u})^2 = (-|\mathbf{k}|\dot{t} + \mathbf{k}\dot{\mathbf{x}})^2 \quad .$$

ku is the product of a lightlike and a timelike vector and therefore negative, so that

$$g_\varepsilon(\mathbf{k};\tau) = e^{i\mathsf{k}\mathsf{x}(\tau)} e^{+\varepsilon\mathsf{k}\mathsf{u}(\tau)} \quad .$$

Obviously, $e^{\varepsilon\mathsf{ku}(\tau)}$ and not simply $e^{-\varepsilon|\mathbf{k}|}$, is the physically motivated regularisating factor resulting from the finite extension of the detector. Using this, we now calculate

$$\begin{aligned}
\int \frac{\mathrm{d}^3k}{|\mathbf{k}|}\, g_\varepsilon(\mathbf{k};\tau) g^*_\varepsilon(\mathbf{k};\tau') &= \int \frac{\mathrm{d}^3k}{|\mathbf{k}|}\, e^{i\mathsf{k}(\mathsf{x}-\mathsf{x}')}\, e^{+\varepsilon\mathsf{k}(\mathsf{u}+\mathsf{u}')} \\
&= \int \mathrm{d}^3k\; e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}'-i\varepsilon(\dot{\mathbf{x}}+\dot{\mathbf{x}}'))}\, \frac{e^{-i|\mathbf{k}|(t-t'-i\varepsilon(\dot{t}+\dot{t}'))}}{|\mathbf{k}|} \\
&= \frac{4\pi}{-(t-t'-i\varepsilon(\dot{t}+\dot{t}'))^2 + (\mathbf{x}-\mathbf{x}'-i\varepsilon(\dot{\mathbf{x}}+\dot{\mathbf{x}}'))^2} \\
+ &= \frac{4\pi}{\left(\mathsf{x}-\mathsf{x}'-i\varepsilon(\dot{\mathsf{x}}+\dot{\mathsf{x}}')\right)^2} \quad .
\end{aligned}$$

From this we finally obtain

$$\langle 0|\phi(\tau)\phi(\tau')|0\rangle = \frac{1/4\pi^2}{\left(\mathsf{x}(\tau)-\mathsf{x}(\tau')-i\varepsilon(\dot{\mathsf{x}}(\tau)+\dot{\mathsf{x}}(\tau'))\right)^2} \quad . \tag{17}$$

This result has to be compared with

$$\langle 0|\phi(\mathsf{x}(\tau))\phi(\mathsf{x}(\tau'))|0\rangle = \frac{-1/4\pi^2}{\left(t(\tau)-t(\tau')-i\varepsilon\right)^2 - \left(\mathbf{x}(\tau)-\mathbf{x}(\tau')\right)^2} \quad , \tag{18}$$

which is obtained from the usual Wightman function (7) by inserting the trajectory $\mathsf{x} = \mathsf{x}(\tau)$. It is remarkable, that (17), in contrast to (18), cannot be written as a function on Minkowski space because it depends on the four-velocity $\dot{\mathsf{x}}$. It can easily be seen, that (18) results from (17) by inserting $\dot{\mathsf{x}}(\tau) = (1,0,0,0)$, which, of course, is inconsistent because the detector is not at rest but moving along the trajectory $\mathsf{x} = \mathsf{x}(\tau)$. As a first test of our new correlation function, we insert all the possible stationary trajectories in Minkowski space, which are classified in [6] into six groups. It turns out that in all six cases the correlation function depends only on $\tau - \tau'$ which has to be expected for a stationary movement. In contrast, inserting into (18) leads to a result which is invariant under time translation only in the cases of the detector at rest and in circular movement. We see this as further evidence that (18) is not suitable for the calculation of the detector response and that (17) should be used instead. Furthermore, (17) is manifestly invariant

under Lorentz transformation of the trajectory, which we consider as another advantage over (18). For example, inserting $t(\tau) = \gamma\tau,\ \mathbf{x}(\tau) = \gamma\mathbf{v}\tau$ with an arbitrary $\mathbf{v}$ satisfying $\mathbf{v}^2 < 1$ leads to the same correlation function (16) as in the case of a detector at rest.

Concluding: Our proposal for calculating the transition rate at time τ for a detector moving on $\mathsf{x}(\tau)$ is

$$\dot{F}_\tau(\omega) = 2\lim_{\varepsilon\to 0}\int_0^\infty \mathrm{d}s\ \mathsf{Re}\left(e^{-i\omega s}\langle 0|\phi(\tau)\phi(\tau - s)|0\rangle\right) \quad , \tag{19}$$

where

$$\langle 0|\phi(\tau)\phi(\tau')|0\rangle = \frac{1/4\pi^2}{\left(\mathsf{x}(\tau) - \mathsf{x}(\tau') - i\varepsilon(\dot{\mathsf{x}}(\tau) + \dot{\mathsf{x}}(\tau'))\right)^2} \quad . \tag{20}$$

Thus, in order to calculate the time-dependent response of a detector in a manifestly causal way, i.e. by using only information on its movements in the *past*, we use a different integration formula and a different correlation function.

It is now easy to show that in the case of an uniformly accelerated trajectory the correlation function is

$$\langle 0|\phi(\tau)\phi(\tau')|0\rangle = \frac{-1/16\pi^2}{\left(\sinh\left(\frac{\tau-\tau'}{2}\right) - i\varepsilon\cosh\left(\frac{\tau-\tau'}{2}\right)\right)^2} \quad ,$$

which clearly is invariant under time translations, as has to be expected from the stationarity of the trajectory. We can thus write the detector response with an integration over the whole real axis, according to (4):

$$\dot{F}_\tau(\omega) = -\frac{1}{16\pi^2}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\mathrm{d}s\ \frac{e^{-i\omega s}}{\left(\sinh\left(\frac{s}{2}\right) - i\varepsilon\cosh\left(\frac{s}{2}\right)\right)^2}$$

and evaluate the integral with the help of contour-integration: We integrate along the real axis in positive direction and along the line $\mathsf{Im}s = 2\pi$ in negative direction. The only singularity within this contour lies at $s = 2i\arctan\varepsilon$. A short calculation shows that because of the periodicity of cosh and sinh, the value $I_\varepsilon(\omega)$ of the integral is connected to the value of the residue by

$$2\pi i\mathsf{Res}\left(\frac{e^{-i\omega s}}{\left(\sinh\left(\frac{s}{2}\right) - i\varepsilon\cosh\left(\frac{s}{2}\right)\right)^2}\ ,\ s = 2i\arctan\varepsilon\right) = I_\varepsilon(\omega) - e^{2\pi\omega}I_\varepsilon(\omega) \quad .$$

The value of the residue is found to be $-4i\omega e^{2\omega\arctan\varepsilon}/1+\varepsilon^2$ which leads to

$$I_\varepsilon(\omega) = \frac{8\pi\omega}{1 - e^{2\pi\omega}}\frac{e^{2\omega\arctan\varepsilon}}{1+\varepsilon^2}$$

and finally

$$\dot{F}_\tau(\omega) = \frac{1}{2\pi}\frac{\omega}{e^{2\pi\omega}-1} \quad .$$

This is of course the well-known thermal Unruh spectrum, which we derived here in a different way as usual, namely in a time-dependent, causal manner and by using a physically founded regularization of the correlation function. This approach may be considered more satisfying than the usual one.

5 Application to a non-stationary example

Let us consider the non-stationary trajectory

$$\begin{aligned} t(\tau) &= \tau - \ln 2 + \sqrt{1+\frac{1}{4}e^{2\tau}} - \ln\left(1+\sqrt{1+\frac{1}{4}e^{2\tau}}\right) \\ x(\tau) &= \frac{1}{2}e^{\tau} \quad , \quad y(\tau) = 0 \quad , \quad z(\tau) = 0 \quad . \end{aligned} \tag{21}$$

This movement interpolates smoothly between rest at $x = 0$ and uniform acceleration along $x^2 - t^2 = 1$. Actually, it is easy to show that $t(\tau) \to \sinh\tau$ and $x(\tau) \to \cosh\tau$ if $\tau \to +\infty$. For $\tau \to -\infty$ we have instead $t(\tau) \to \tau + 1 - 2\ln 2$ and $x(\tau) \to 0$. The square of the four-acceleration

$$\mathsf{b}^2 = -\ddot{t}^2 + \ddot{x}^2 = \frac{1}{1+4e^{-2\tau}}$$

shows that the proper acceleration $\alpha = \sqrt{\mathsf{b}^2}$ is indeed smoothly increasing from 0 to 1. The radiation spectrum is now time-dependent, of course. The result of a numerical calculation for three times $\tau = -4, 1, 6$ is shown in figure 2. At time $\tau = -4$ the movement has not really yet begun, so one gets the spectrum of the detector at rest. At time $\tau = 6$ the movement ran a fairly long time with approximately constant acceleration $\alpha = 1$, so the thermal spectrum results, which does not change any more at later times. It is remarkable, that at the intermediate time $\tau = 1$, i.e. $\alpha = 0.65$, the transition-rate decays *slower* for high ω than in the case of the thermal spectrum at late times. This means, that the increasing of acceleration takes place smoothly but not adiabatically: The radiation spectrum is of a non-thermal nature, and not simply thermal with only a time-varying temperature. This shows that not only acceleration contributes to the radiation effect, but higher time derivatives as well.

6 Conclusion

Starting with the intention to derive an explicitly causal formulation for the transition rate of a non-inertial Unruh-DeWitt detector, we were led to the

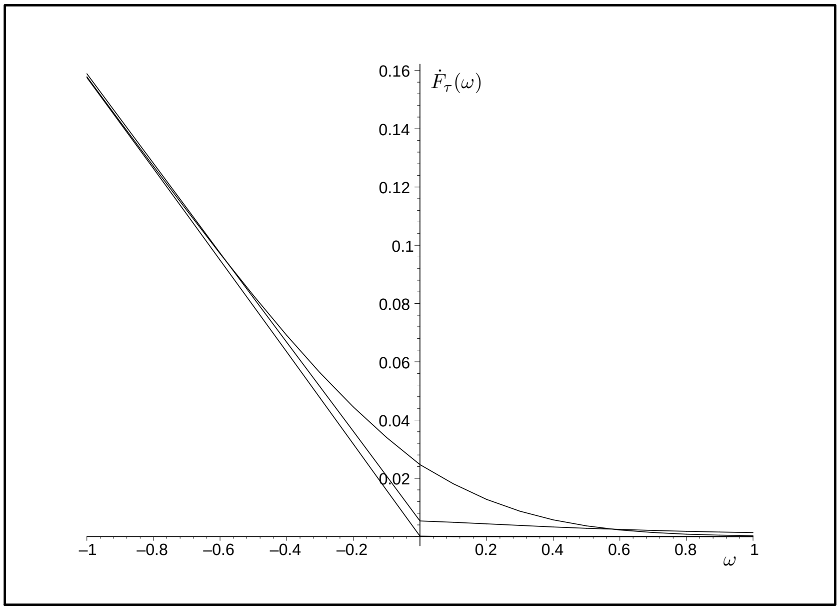

Figure 2: Non-stationary Unruh-like spectrum

question of which regularization of the scalar field's correlation function is appropriate to this problem. The usual prescription $t \to t - i\varepsilon$ or the equivalent insertion of the convergence factor $e^{-\varepsilon|\mathbf{k}|}$ into the Fourier representation was shown to be inappropriate by a numerical calculation, which lead to a time-dependent unphysical result in the case of uniformly accelerated motion. Following Takagi we arrived at the correct regularization by considering the pointlike detector as the limit of a finite, rigid detector - where rigidity is defined with respect to its own momentary rest system. The result is

$$\dot{F}_\tau(\omega) = 2 \lim_{\varepsilon \to 0} \int_0^\infty \mathrm{d}s \; \mathsf{Re} \left(e^{-i\omega s} \langle 0|\phi(\tau)\phi(\tau - s)|0\rangle \right) \quad ,$$

$$\langle 0|\phi(\tau)\phi(\tau')|0\rangle = \frac{1/4\pi^2}{\big(\mathsf{x}(\tau) - \mathsf{x}(\tau') - i\varepsilon(\dot{\mathsf{x}}(\tau) + \dot{\mathsf{x}}(\tau'))\big)^2} \quad .$$

We consider this as the physically well-founded form of the response function of an infinitesimal detector.

Finally, a short criticism of an often heared argument regarding the Unruh effect may be formulated. The fact that Unruh radiation is exactly *thermal* is astonishing and demands for some deeper explanation. A well-known argument is based on the existence of an acceleration horizon which seperates the detector from some degrees of freedom of the scalar field.

Indeed, the eternal uniformly accelerating detector or observer is causally separated from the "left half" of Minkowski space by the horizon $t = x$.

Therefore, as the argument goes, the observer describes the global vacuum state $|0\rangle$ of the scalar field as a density matrix which results from tracing out the degrees of freedom behind the horizon. This density matrix turns out to be thermal, with a temperature of $T = \alpha/2\pi$, cf. [7]. In this way the ocurrence of the thermal spectrum can be understood as a consequence of the existence of an acceleration horizon and the associated "hiding" of degrees of freedom. This may look convincing, but is not without problem if considered from a causal viewpoint: On the one hand, the existence of an acceleration horizon can be stated only if the *whole course* of the trajectory is known, in particular the future behaviour up to $t = +\infty$. The existence of a thermal spectrum, on the other hand, is a fact which results only from the *past* movement of the detector. Then the problem is posed, that an observer, who percives thermal radiation here and now, *cannot* explain this by showing up the acceleration horizon, because this is not an observable border, existing somewhere in spacetime, but rather an abstract object which can be constructed only "posthumously" after the entire movement is known. At any finite time, there exist for the accelerated observer no more and no less unobservable degrees of freedom as for the observer at rest.

The problem which - in my opinion - really calls for an explanation, is the fact, that an observer moving along trajectory (21), which smoothly goes over from rest to an arbitrarily long period of almost uniform acceleration, sees a radiation spectrum which is thermal to any arbitrary precision - without making any assumptions on the future movement of the detector. Of course, the trajectory (21) is defined for all t, and with it comes the acceleration horizon $t = x$, but this does not enter the calculation. The spectrum measured at time τ would not change at all if the detector returns to rest afterwards. Thus we have the following situation: A detector which is asymptotically at rest for $t \to \pm\infty$, is moving for an arbitrarily long (but finite) time with almost uniform acceleration and percives an (almost) thermal radiation-spectrum. Because he returns from asymptotic rest to asymptotic rest, there is *no* acceleration horizon in this case, and *no* hidden degree of freedom. The question for a simple explanation of the (arbitrarily precise) thermality of the observed radiation is but as important in this case as in the idealized case of eternal hyperbolic movement.

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